THE PROBLEM OF THE BOUNDARY LAYER AT A THERMALLY ISOLATED PLATE

V. I. Voronin and A. E. Blazhkov

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New solutions are given for the energy equation of a compressible boundary layer at a plate in absence of heat transfer.

In [1] was obtained a more general solution of the energy equation for an incompressible boundary layer when Pr = 1 and λ and μ are constant, which differs from the familiar Crocco integral by the additional term $C\partial u/\partial y$. Assuming that the gas is incompressible inside the boundary layer, this solution nevertheless satisfies the condition $i = i_\infty$ at the outer boundary even when there is a pressure gradient for which there is no physical basis. In addition, when the gas is incompressible the viscosity coefficient depends significantly on the temperature. Hence it is appropriate to generalize the solution obtained in [1] to the problem of the flow of a compressible gas past a plate.

For $\mu = \mu_0(T/T_0)$ and Pr = 1 the equations for the compressible boundary layer in the variables u and ξ take the form

$$Z^2 \frac{\partial^2 Z}{\partial u^2} = \nu_0 u \frac{\partial Z}{\partial \xi}, \quad (1)$$

$$Z^2 \frac{\partial^2 i}{\partial u^2} = \nu_0 u \frac{\partial i}{\partial \xi}, \quad (2)$$

where $Z = \nu_0(\partial u/\partial \eta)$ is the "distorted" friction stress; $i = c_p T + u^2/2$ is the enthalpy of the gas. Equation (1) can be solved independently of (2). It is easy to see that the solution of Eq. (2) is

$$i = aZ + bu + c. \quad (3)$$

For an impermeable, thermally-isolated plate b = 0 and as $\xi \to \infty$, Eq. (3) becomes Crocco's solution. Using the familiar solution due to Blasius, $Z = \sqrt{\nu_0 u_\infty^3/\xi} f''$, we find an expression for the temperature

$$T = \frac{a}{c_p}\sqrt{\frac{\nu_0 u_\infty^3}{\xi}} f'' - \frac{u^2}{2c_p} + T_0.$$

The constant a can be determined by specifying the temperature of the plate near the leading edge. On Fig. 1 are given graphs of the temperature distributions and thermal fluxes for M = 4 and T = 500°K when ξ = 10 cm. Thus, when there is no heat transfer at the plate, Eq. (2) is satisfied by two solutions: Crocco's integral and (3). We can assume that there is a whole set of similar solutions. We attempt to find them by an approximate method. If we substitute $Z = \sqrt{\nu_0 u_\infty^3/\xi} f''$ in (2) and write it in nondimensional variables $\bar{u} = u/u_\infty$, $\bar{\xi} = \xi/l$

$$f''^2 \frac{\partial^2 i_1}{\partial \bar{u}^2} = \bar{u}\,\bar{\xi} \frac{\partial i_1}{\partial \bar{\xi}}, \text{ where } i_1 = i - i_\infty. \quad (4)$$

Further, in (4) we put $f'' \approx 0.332\sqrt{1-\bar{u}^3}$ and introduce the variable $t = \bar{u}^3$

$$t(1-t)\frac{\partial^2 i_1}{\partial t^2} + \frac{2}{3}(1-t)\frac{\partial i_1}{\partial t} = \bar{\xi}\frac{\partial i_1}{\partial \bar{\xi}}. \quad (5)$$

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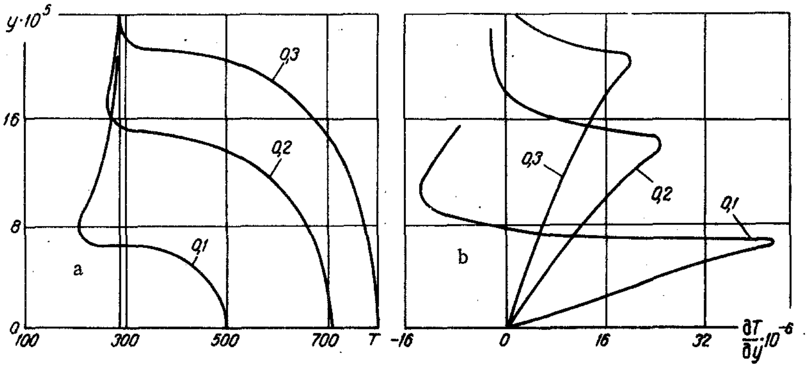

Fig. 1. Graphs of the temperature distributions (*a*) and the heat fluxes (b) in the boundary layer of a plate for various values of $\bar{\xi}$, indicated by the numbers attached to the curves (M = 4; T = 500°K; ξ = 10 cm; y, m; $\partial T/\partial y$, deg/m).

After separating the variables we obtain the equations

$$t(1-t)Q''(t)+\frac{2}{3}(1-t)Q'(t)-kQ(t)=0, \tag{6}$$

$$\frac{\psi'(\bar{\xi})}{\psi(\bar{\xi})}=\frac{k}{\bar{\xi}}. \tag{7}$$

The solution of (7) is $\psi(\bar{\xi}) = \bar{\xi}^k$. The hypergeometric equation (6) has the solution [2]

$$Q(t)=AF(\alpha,\ \beta,\ \gamma;\ t)+Bt^{1-\gamma}F(\alpha_1,\ \beta_1,\ \gamma_1;\ t) \tag{8}$$

The parameters of the hypergeometric functions are:

$$\alpha=\frac{-1\pm\sqrt{1-36k}}{6};\quad \beta=\frac{-1\mp\sqrt{1-36k}}{6};\quad \gamma=\frac{2}{3};$$
$$\alpha_1=1-\gamma+\alpha;\quad \beta_1=1-\gamma+\beta;\quad \gamma_1=2-\gamma. \tag{9}$$

The solution of Eq. (5) must satisfy the boundary conditions

$$i_1=0\quad \text{for}\quad \bar{u}=1,$$
$$\frac{\partial i_1}{\partial \bar{u}}=0\quad \text{for}\quad \bar{u}=0.$$

It follows from the last condition that B = 0. From the condition at the outer boundary of the layer we find that

$$AF(\alpha,\ \beta,\ \gamma;\ 1)=0.$$

If a nontrivial solution is to exist it is necessary that A ≠ 0. Then $F(\alpha, \beta, \gamma; 1) = 0$. Since

$$F(\alpha,\ \beta,\ \gamma;\ 1)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)},$$

either $\Gamma(\gamma-\alpha) = \infty$ or $\Gamma(\gamma-\beta) = \infty$. Without loss of generality we can assume that $\Gamma(\gamma-\alpha) = \infty$. Hence $\gamma-\alpha = 0, -1, -2, \ldots, -n$. From this we obtain the equation

$$k=-\frac{(1+n)(2+3n)}{3}.$$

The required solution takes the form

$$i=i_\infty+A\frac{F(\alpha,\ \beta,\ \gamma;\ t)}{\bar{\xi}^{-k}}.$$

The following linear superposition is also a solution of (5):

$$i=i_\infty+\sum_{n=0}^{\infty}A_k\frac{F(\alpha_k,\ \beta_k,\ \gamma;\ t)}{\bar{\xi}^{-k}},\qquad k=k(n).$$

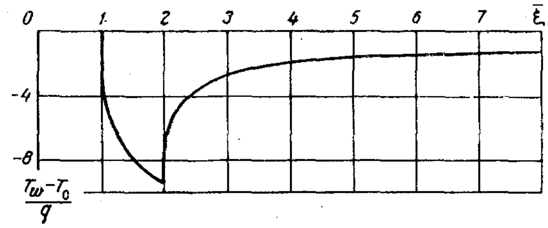

Fig. 2. Graph of the temperature of the plate as a function of $\bar{\xi}$ when there is a zone $1 < \bar{\xi} < 2$ with heat exchange.

Thus, we have a whole spectrum of "nonheat-conducting" solutions ($n = 0, 1, 2, \ldots$). These solutions can easily be generalized using Mangler's transformation to axi-symmetric boundary layers. Thus, for the gradientless flow of a compressible gas round a cone, Eq. (3) takes the form

$$i = a\sqrt{3}\,\frac{Z}{\xi} + c_p T_0.$$

In spite of the fact that $\bar{\xi} = 0$, the solutions we have found have the feature that they are of interest from the point of view of the possible occurrence of a thermally-isolated surface, and a fortiori that the singularity at the vertex of the plate (cone) has no effect on the correctness of the solutions of the equations of the boundary layer.

As a physical example, corresponding to the solutions given above, we consider the problem of the thermal boundary layer of a plate for which at a distance l from the leading edge there is a zone $1 < \bar{\xi} < \bar{\xi}_2$ with heat exchange:

$$\left(\frac{\partial i_1}{\partial \bar{u}}\right)_{\bar{u}=0} = c_p q = \text{const}, \text{where } q = \frac{u_\infty}{\tau_w}\,\frac{\lambda}{c_p}\,\frac{\partial T}{\partial y}.$$

The remainder of the plate is assumed to be thermally isolated. We assume that Crocco's solution $i = i_\infty$ is valid for the initial segment l. The problem is to determine the solution of Eq. (5) for $1 < \bar{\xi} < \infty$. We seek the solution by an operational method. We introduce the variable $s = \ln \bar{\xi}$ into (5) and make a Laplace transformation with respect to s:

$$t(1-t)\frac{d^2 i_1^*}{dt^2} + \frac{2}{3}(1-t)\frac{d i_1^*}{dt} - p i_1^* = -p i_1^*(s=0). \tag{10}$$

The initial and boundary conditions are

$$i_1^* = 0 \quad \text{for} \quad s = 0,$$
$$i_1^* = 0 \quad \text{for} \quad \bar{u} = 1,$$
$$\frac{\partial i_1^*}{\partial \bar{u}} = c_p q\,[1 - \exp(-p s_2)] \quad \text{for} \quad \bar{u} = 0. \tag{11}$$

Substituting the initial condition in (10), we obtain an equation, similar to (6), which has the solution

$$i_1^* = AF(\alpha, \beta, \gamma;\ t) + Bt^{1-\gamma}F(\alpha_1, \beta_1, \gamma_1;\ t). \tag{12}$$

Here the parameters of the hypergeometric functions are given by (9) if k is replaced by p. The last boundary condition of (11) allows us to determine

$$B = c_p q\,[1 - \exp(-p s_2)].$$

From the condition at the outer edge of the layer we find that

$$A = -B\frac{F(\alpha_1, \beta_1, \gamma_1;\ 1)}{F(\alpha, \beta, \gamma;\ 1)} = \frac{\Gamma(\gamma_1)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)\,\Gamma(\gamma_1-\alpha_1)\,\Gamma(\gamma_1-\beta_1)}\,c_p q\,[1 - \exp(-p s_2)].$$

Using the functional relations for the γ-function

$$\Gamma(\gamma-\alpha) = \Gamma(\beta+1) = \beta\Gamma(\beta),$$
$$\Gamma(\gamma-\beta) = \Gamma(\alpha+1) = \alpha\Gamma(\alpha),$$
$$\Gamma(\gamma_1-\alpha_1) = \Gamma(1+\beta_1) = \Gamma(1-\alpha) = \frac{\pi}{\Gamma(\alpha)\sin\pi\alpha},$$
$$\Gamma(\gamma_1-\beta_1) = \Gamma(1+\alpha_1) = \Gamma(1-\beta) = \frac{\pi}{\Gamma(\beta)\sin\pi\beta},$$

we can transform the expression for A to the form

$$A = -\frac{\Gamma(\gamma_1)}{\Gamma(\gamma)\pi^2}\alpha\beta\Gamma^2(\alpha)\Gamma^2(\beta)\sin\pi\alpha\sin\pi\beta\, c_p q[1-\exp(-ps_2)].$$

Substituting A and B in (12) we obtain the solution in the transformed space. The original is

$$i_1 = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} -\frac{\Gamma(\gamma_1)}{\Gamma(\gamma)\pi^2}\alpha\beta\Gamma^2(\alpha)\Gamma^2(\beta)\sin\pi\alpha\sin\pi\beta\, c_p q[1-\exp(-ps_2)]\times F(\alpha,\ \beta,\ \gamma;\ t)\exp(ps)\frac{dp}{p}$$
$$+\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} c_p q[1-\exp(-ps_2)]\, t^{1-\gamma}\exp(ps)F(\alpha_1,\ \beta_1,\ \gamma_1;\ t)\frac{dp}{p}. \quad (13)$$

It is easy to establish that the integrand of the first term is a meromorphic function with simple poles $\alpha_n = -n$ or $\beta_n = -n$. Putting $\alpha_n = -n$, from (9) we find that $p_n = -[(6n-1)^2 - 1]/36$, $\beta_n = (3n-1)/3$. Hence, the first term can be replaced by the sum of the residues at the poles p_n and the solution of (13) is transformed to the form

$$i - i_\infty = -\frac{c_p q\pi\sqrt{3}}{6}\sum_{n=1}^{\infty}\frac{6n-1}{(n!)^2}\Gamma^2\left(\frac{3n-1}{3}\right)\times\left[1-\delta\bar{\xi}_2^{\frac{(6n-1)^2-1}{36}}\right]\frac{F(\alpha_n,\ \beta_n,\gamma_n;\ t)}{\bar{\xi}^{\frac{(6n-1)^2-1}{36}}} + I_2, \quad (14)$$

where

$$\delta = \begin{cases} 1 & \text{for } \bar{\xi} > \bar{\xi}_2, \\ 0 & \text{for } \bar{\xi} < \bar{\xi}_2, \end{cases}$$

where I_2, the second term in (13), is the integral of an entire function and vanishes when $\bar{u} = 0$. In (14) $F(\alpha_n, \beta_n, \gamma_n;\ t)$ vanishes when $t = 1$ and, in addition, $(\partial F/\partial\bar{u})_{\bar{u}=0} = 0$. Thus, (14) is the sum of two terms, the first of which

$$I_1 = -\frac{c_p q\pi\sqrt{3}}{6}\sum_{n=1}^{\infty}\frac{6n-1}{(n!)^2}\Gamma^2\left(\frac{3n-1}{3}\right)\times\left[1-\delta\bar{\xi}_2^{\frac{(6n-1)^2-1}{36}}\right]\frac{F(\alpha_n,\ \beta_n,\ \gamma_n;\ t)}{\bar{\xi}^{\frac{(6n-1)^2-1}{36}}}$$

is the linear superposition of approximate "nonheat-conducting" solutions, previously discussed. For the temperature at the plate ($\bar{u} = 0$) we obtain the expression

$$T_w - T_0 = -\frac{q\pi\sqrt{3}}{6}\sum_{n=0}^{\infty}\frac{6n-1}{(n!)^2}\Gamma^2\left(\frac{3n-1}{3}\right)\times\left[1-\delta\bar{\xi}_2^{\frac{(6n-1)^2-1}{36}}\right]\frac{1}{\bar{\xi}^{\frac{(6n-1)^2-1}{36}}}. \quad (15)$$

We see from (15) that the temperature of the plate in this problem is determined only by a linear superposition of appropriate "nonheat-conducting" solutions. On Fig. 2 is shown a graph of $(T_w - T_0)/q$ as a function of $\bar{\xi}$ for q = const.

NOTATION

x, y	are the longitudinal and transverse coordinates;
ξ, η	are the Dorodnitsyn variables;
u, v	are the longitudinal and transverse velocity components;
a, b, c	are constants;
ϑ	is the heat-transfer coefficient;
μ	is the dynamic viscosity coefficient;
ν	is the kinetic viscosity coefficient;
l	is a characteristic dimension of the body;
T	is the absolute temperature;
Γ	is the Euler γ-function;
n	is a positive integer;
c_p	is the specific heat capacity of the gas;
λ	is the thermal conductivity coefficient;
j	is the imaginary unit, $\sqrt{-1}$;

τ is the friction stress;

M is the Mach number.

Subscripts

w is for parameters at the wall;

0 is for gas parameters for adiabatic drag;

∞ is for gas parameters in the undisturbed flow.

LITERATURE CITED

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